Deformed Free Particle as a Pulsed Oscillator

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Abstract

The periodically pulsed oscillator (with period T) is shown to share certain common properties with the q-deformed free particle. The two systems are characterized by a common deformation parameter q, and reduce to a usual free particle as q tends to unity (the boson limit). For the deformed free particle, q is a real number, whereas for the pulsed oscillator it belongs to S^1. The propagator for the Chebyshev process is also derived from which the propagator for the T-evolution of the deformed free particle is obtained.

1 Introduction

As is well-known, the damped harmonic oscillator can be seen as a harmonic oscillator via a scale transformation (see, e.g., [1]). In much the same fashion, the harmonic oscillator may be conceived as a free particle via the de Alfaro-Fubini-Furlen-Jackiw transformation [2],

$$\mathbf{r}(t) = \mathbf{R}(s)\sec(\omega s) \qquad t = \tan(\omega s) \tag{1}$$

where s is a new time-like parameter. In this paper, we argue in a slightly different manner that a q-deformed free particle may be viewed as the pulsed harmonic oscillator in a generalized sense.

By a q-deformed free particle (or a q-free particle), we mean, as is described in Section 2, a particle which obeys the q-counterpart of Newton's force-free equation of motion,

$$D^2_{s;q}y(s) = 0, \tag{2}$$

or equivalently the q-difference equation,

$$q^{-1}y(q^2s) - (q+q^{-1})y(s) + qy(q^{-2}s) = 0, \tag{3}$$

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where s is a time parameter and q is the deformation parameter. The q-free particle becomes the usual Newtonian free particle only when $q \to 1$.

The pulsed harmonic oscillator (p-oscillator) is not identical to the so-called kicked harmonic oscillator that is a harmonic oscillator subjected to periodic kicks. It is a free particle subject to the periodic pulses of the Hookes force, details of which are discussed in Section 3. Right after the mth pulse, that is, when $t \to mT$, it obeys the difference equation

$$x(t+T) - (2 - \omega^2 T^2)x(t) + x(t-T) = 0 \tag{4}$$

where ω is a constant and T is the period of pulses.

We observe that the above two difference equations at a fixed time can be reduced to the recursion relation for the Chebyshev polynomials and that both the q-free particle and the p-oscillator can be characterized by a common deformation parameter q. Section 4 will deal with a generic q-object (a generalized p-oscillator) unifying the two systems under the condition $q + q^{-1} \in \mathbf{R}$. In Section 5, we derive by path integration the q-dependent propagator for the q-object. In Section 6, the propagators for the two systems are distinguished by the values of q; the former has $q \in \mathbf{R}$, while the latter has $q \in S^1$. The caustics and the harmonic oscillator limit are also characterized by particular values of the deformation parameter q.

2 The q-Deformed Free Particle

We begin with a short review of q-deformation (for a more detailed review, see, e.g., [3]). Let N, a and $a^\dagger$ be the linear operators acting on the Fock space $\mathcal{F} = \{|n\rangle : \ n \in \mathbf{N}\}$ as

$$N|n\rangle = n|n\rangle, \qquad a|n\rangle = \sqrt{[n]}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle \tag{5}$$

with $a|0\rangle = 0$. In the above, we have used the symmetrically q-deformed number,

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{6}$$

where q is a fixed real or complex parameter. The operators satisfy the commutator,

$$aa^\dagger - qa^\dagger a = q^{-N}, \tag{7}$$

which has been used for the q-deformed harmonic oscillator by Macfarlane [4] and by Biedenharn [5]. In the limit that $q \to 1$, the $[n]$ tends to the ordinary number n, and the above linear operators become those for the well-known harmonic oscillator or the boson operators. If we introduce another set of operators $\{c, \bar{c}\}$ by $c = q^{N/2}a$ and $\bar{c} = a^\dagger q^{N/2}$, the commutator (7) can be written as

$$c\bar{c} - q^2 \bar{c}c = 1. \tag{8}$$

If q^2 is given as roots of unity, say, $q = \exp[\pi i/(s+1)]$ $(s \in \mathbf{N})$, then the Fock space becomes finite dimensional (of $s+1$ dimensions). In particular, if $s = 1$, then $q^2 = -1$, and the linear operators c and $\bar{c}$ behave like fermion operators. The creation operator $a^\dagger$ is the hermitian conjugate of the annihilation operator a if the q-deformed numbers $[n]$ are all real. The deformed numbers are real if $q \in \mathbf{R}$ or $q \in S^1$, that is, if q is real or given by $q = e^{i\theta}$ with a real number θ. If $a^\dagger$ is the hermitian conjugate of a, then $\bar{c}$ cannot be the hermitian operator of c unless q is chosen to be a real number.

The q-derivative (or the symmetric Jackson derivative) of a function $f(s)$, corresponding to the deformation (7), is defined by

$$D_{s;q}f(s) = \frac{f(qs) - f(q^{-1}s)}{(q - q^{-1})s}. \tag{9}$$

The q-counterpart of Newton's equation in one dimension is

$$D^2_{s;q}y(s) = F(y) \tag{10}$$

where $F(y)$ is a force exerted to the system.

What we are referring to as a q-deformed free (q-free) particle is the system obeying the force-free equation ($F(y) = 0$); that is, the difference equation (3). The usual free particle solution,

$$y(s) = as + c \qquad (a,\, c : \text{constant} \in \mathbf{R}), \tag{11}$$

is indeed the solution of (3). Insofar as the time parameter s changes translationally and uniformly, the q-free particle is nothing more than the ordinary free particle. Yet, the q-free particle equation (3) differs from the difference equation corresponding to the usual force-free Newtonian equation,

$$x(t+T) - 2x(t) + x(t-T) = 0, \tag{12}$$

which is a special case of the difference equation for the p-oscillator (4) with $\omega = 0$ (where T is no longer the pulsing period but any finite time interval). The solution of (12), $x(t) = at + b$, is the same as (11) in form, but not in content unless q tends to unity. This is because the q-free particle equation (3) reduces to the Newtonian form (12) only in the limit $q \to 1$.

The difference equation (12) dictates the time-evolution of the particle under the discrete time-translation $t - T \to t \to t + T$, whereas the deformed difference equation (3) stipulates the progression of the particle under the time-scaling $q^{-2}s \to s \to q^2 s$. In fact, the time transformation [6]

$$s = q^{2t/T} \tag{13}$$

with $q \neq 0$ relates the time-translation to the time-scaling as

$$qs = q^{2(t+T/2)/T}, \tag{14}$$

and the q-derivative to the ordinary discrete derivative as

$$\frac{y(qs) - y(q^{-1}s)}{(q - q^{-1})s} = \frac{T}{q - q^{-1}} \frac{y(t+T/2) - y(t-T/2)}{T}, \tag{15}$$

or more formally

$$D_{s;q}y(s) = \frac{T}{q-q^{-1}}\Delta_{t;T}y(t). \tag{16}$$

Therefore the trajectory of a q-free particle evolving with the Newtonian time t is given by

$$y(t) = aq^{2t/T} + c. \tag{17}$$

In the transformation (13), q may be a complex number and the resulting value of s can be complex as well. However, for a real trajectory of a q-free particle, as described by (17), q must be a real number.

Next we focus our attention on the q-progression of the q-free particle. First, considering m times of the q-progression from a fixed value s_0 (e.g., $s_0 = 1$ corresponding to $t = 0$) of s, we substitute $s = q^{2m}s_0$ into (3) to obtain

$$q^{-1}y(q^{2m+2}s) - (q+q^{-1})y(q^{2m}s) + qy(q^{2m-2}s) = 0. \tag{18}$$

Then we let

$$u_m(q) = y(q^{2m}s_0)/q^m \tag{19}$$

to rewrite () in the form,

$$u_{m+1}(q) - (q+q^{-1})u_m(q) + u_{m-1}(q) = 0, \tag{20}$$

which is the q-progress equation of the deformed free particle. A general solution of this equation is

$$u_m(q) = bq^m + cq^{-m} \qquad (b,\ c:\ \text{constant}).$$

Correspondingly,

$$y(q^{2m}s_0) = aq^{2m}s_0 + c, \tag{21}$$

where b is chosen to be as_0. This solution for the mth progression is of course directly obtainable from (17) by letting $t = t_0 + mT$ with $s_0 = e^{2t_0/T}$. More generally, replacing s_0 by s, we can write (22) as

$$y(q^{2m}s) = aq^{2m}s + c. \tag{22}$$

It is also evident that the corresponding mth step momentum,

$$p_y(q^{2m}s) = MD_{s;q}y(q^{2m}s) = Maq^{2m}, \tag{23}$$

where M is the mass of the particle, satisfies, if divided by q^m, the same q-progress equation (20).

Noticeably, the q-progress equation (20) is the recursion relation for the Chebyshev polynomials of type I and type II (see, e.g., [7]) given, respectively, by

$$T_m[\cos\varphi] = \cos m\varphi \qquad U_m[\cos\varphi] = \frac{\sin(m+1)\varphi}{\sin\varphi} \tag{24}$$

when $q = e^{-i\varphi}$ ($\varphi \in \mathbf{R}$). The solution of (20), satisfying the boundary conditions $u_m(1) = 0$ and $u_1(q) = 1$, is indeed the deformed number $[m]$ (which is the Chebyshev polynomial of type II if $q = e^{-i\varphi}$). We shall refer to an evolution obeying the recursion relation (20) as a Chebyshev process.

3 The Pulsed Oscillator

The pulsed oscillator (p-oscillator) is a free particle which undergoes periodic pulses of Hooke's force $F(t) = -M\omega^2 x\,\delta(t/T - m)$ where $M\omega^2$ is Hooke's constant, T is the period of pulses and $m \in \mathbf{Z}$. The Lagrangian is given by

$$L = \frac{1}{2}M\dot{x}^2 - \sum_m \frac{1}{2}M\omega^2 T x^2 \delta(t - t_m) \tag{25}$$

where $t_m = mT$. Hooke's force is exerted not continuously but periodically and instantaneously at $t = t_m$. During the period between two consecutive pulses, the system is a free particle.

The action integral for a time interval $\tau = t'' - t$" is

$$S(t'', t') = \int_{t'}^{t''} \left[\frac{1}{2}M\dot{x}^2 - \frac{1}{2}M\omega^2 T x^2 \sum_m \delta(t - t_m)\right] dt. \tag{26}$$

The action integral for a short time interval $\tau_j = t_j - t_{j-1} \ll T = t_m - t_{m-1}$ may be chosen as

$$\begin{aligned} S_j &= \int_{t_{j-1}}^{t_j} \left[\frac{1}{2}M\dot{x}^2 - \frac{1}{4}M\omega^2 T x^2 \{\delta(t - t_m) + \delta(t - t_{m-1})\}\right] dt, \qquad (27)\\ &\doteq \frac{M}{2\tau_j}(x_j - x_{j-1})^2 - \frac{1}{4}M\omega^2 T\,\{x_m^2\,\delta(m, j) + x_{m-1}^2\,\delta(m-1, j)\}, \end{aligned}$$

where $x_k = x(t_k)$ and

$$\begin{aligned} \delta(k, j) &= 1 \quad \text{if } \; t_{j-1} < kT < t_j \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Naturally the action evaluated along the classical path from $t = t_{m-1} + \epsilon$ to $t = t_m - \epsilon$ $(0 < \epsilon \ll T)$ without involving pulses yields that of the free particle:

$$S_0(t_m, t_{m-1}) = \lim_{\epsilon \to 0} S(t_m + \epsilon, t_{m-1} - \epsilon) = \frac{M}{2T}(x_m - x_{m-1})^2. \tag{28}$$

The action over one period, involving pulses, becomes

$$S(t_m, t_{m-1}) = \lim_{\epsilon \to 0} S(t_m + \epsilon, t_{m-1} - \epsilon) = \frac{M}{2T}(x_m - x_{m-1})^2 - \frac{1}{4}M^2 T(x_m^2 + x_{m-1}^2). \tag{29}$$

This is symmetric with respect to x_m and x_{m-1}. Although we can construct nonsymmetric one-period actions, we employ the symmetrized action (29) for convenience.

Calculating the canonical momenta from the symmetrized action (29) by

$$\begin{aligned} p_m &= \partial\bar{S}/\partial x_m &&= (M/T)(x_m - x_{m-1}) - (M\omega^2 T/2)x_m \\ p_{m-1} &= -\partial\bar{S}/\partial x_{m-1} &&= (M/T)(x_m - x_{m-1}) + (M\omega^2 T/2)x_{m-1} \end{aligned}$$

we find the area-preserving linear map in phase space:

$$\begin{aligned} x_m &= x_{m-1} + (T/2M)(1 - \tfrac{1}{4}\omega^2 T^2)^{-1}\ (p_m + p_{m-1}) \\ p_m &= p_{m-1} - (M\omega^2/2)T(x_m + x_{m-1}) \end{aligned} \tag{30}$$

The evolution of the classical trajectory in phase space obeying the linear map (30) is not chaotic, and may not particularly be interesting for the study of the relation between chaos and quantum recurrences [8]. Nonetheless, it is interesting that both x_m and p_m in (30) obey the well-known recursion relation for the Chebyshev polynomials,

$$u_{m+1}(z) - 2z\, u_m(z) + u_{m-1}(z) = 0, \tag{31}$$

when the following identification is made;

$$z = 1 - \frac{1}{2}\omega^2 T^2. \tag{32}$$

With $z = \cos\varphi$, the solutions of the recursion relation (31) are given in terms of the Chebyshev polynomials of (24). If $0 < \omega^2 T^2 < 4$, then $\varphi \in \mathbf{R}$. Hence the classical discrete solutions for $x(t)$ and $p(t)$ oscillate sinusoidally, which are indeed physical solutions for the proper p-oscillator. If $\omega^2 T^2 < 0$ or $4 < \omega^2 T^2$, then φ has to be complex; so the solutions of (31) are not oscillatory and do not physically represent the p-oscillator. Nevertheless we may handle the physically proper solutions and the physically improper solutions together as solutions of the p-oscillator in a generalized sense.

It should be noted that although the area-preserving linear maps obtainable from the nonsymmetric one-period actions differ in form from (30) each of x_m and p_m resulting from the nonsymmetric actions also satisfies the same recursion relation (31).

4 q-Objects

It is remarkable that the recursion relation for the p-oscillator (31) coincides with the q-progress equation (20) for the q-free particle if $q = e^{-i\varphi}$. Apparently the time-evolution of both systems is basically the Chebyshev process. While the q-difference equation for the q-free particle takes the Newtonian form in the limit $q \to 1$, the p-oscillator approaches the free particle in the limit $e^{-i\varphi} \to 1$ $(\omega \to 0)$. Therefore, by relating q to φ by $q = e^{-i\varphi}$, we should be able to treat both the q-free particle and the p-oscillator in a unified manner. In other words, we may consider the two systems as special cases of a generic q-object with $q = e^{-i\varphi}$.

The generic q-object may be defined with a non-zero complex valued q. However, we restrict ourselves to the case where $z = \cos\varphi = (q + q^{-1})/2$ is real. Under this condition, $q \in \mathbf{R}$ or $q \in S^1$. In fact, such a q-object is equivalent to the generalized p-oscillator possessing the proper and improper solutions. Therefore, it is convenient to utilize the oscillator's frequency ω as a parameter even for the q object. We then put solutions of the q-object into three classes as follows:

$$\begin{array}{llll} \text{(i)} & 0<\omega^2T^2<4; & \varphi=\cos^{-1}(1-\omega^2T^2/2)\in\mathbf{R}; & q\in S^1 \\ \text{(ii)} & \omega^2T^2<0; & i\varphi=\cosh^{-1}(1+|\omega|^2T^2/2)\in\mathbf{R}; & q\in\mathbf{R}^+ \\ \text{(iii)} & 4<\omega^2T^2; & i\varphi=i\pi+\cosh^{-1}(\omega^2T^2/2-1)\in\mathbf{R}; & q\in\mathbf{R}^- \end{array}$$

Evidently, case (i) corresponds to the proper p-oscillator. As has been mentioned earlier, for the real trajectory (17) of the q-free particle, q must be real. However, for a continuous evolution with the Newtonian time t, $y(t)$ can be real only when q is positive. Hence, case (ii) should correspond to the (proper) evolution of the q-free particle. As each discrete translation of time by T causes the scaling of s by q^2, y_m remains to be real even if q is a negative real number provided s_0 is real. For a continuous evolution with a negative q, $y(t)$ takes complex values in general. Thus, the discrete evolution of the hopping q-free particle belongs to case (iii). In this manner, the q-free particle may be viewed as a form of the improper p-oscillator.

5 The Propagator for the Chebyshev Process

In what follows we calculate the propagator for the p-oscillator, and see how it depends on the deformation parameter q. Then we interpret it more generally as the propagator (in the T-evolution) for the generic q-object obeying the Chebyshev process.

The propagator for the system with the Lagarangian (25) can be calculated from Feynman's path integral,

$$K(x'',x';\tau)=\lim_{N\to\infty}\int_{x'=x(t_0)}^{x''=x(t_N)}\prod_{j=1}^{N}\exp\left[\frac{i}{\hbar}S_j\right]\prod_{j=1}^{N}\left[\frac{M}{2\pi i\hbar\tau_j}\right]^{1/2}\prod_{j=1}^{N-1}dx_j, \tag{33}$$

where $\tau=t_N-t_0$ is a fixed total time interval. The propagator is to have the following properties:

$$K(x'',t'';x',t')=\int K(x'',t'';x,t)\,K(x,t;x',t')\,dx(t), \tag{34}$$

$$\lim_{t''\to t'}K(x'',t'';x',t')=\delta(x''-x'). \tag{35}$$

To calculate the one-period propagator, we first put it by making use of the property (34) in the form

$$\begin{aligned} &K(x_m,x_{m-1};T)= \\ &\lim_{\epsilon\to 0}\int K(x'',t_m+\epsilon;x_2,t_m-\epsilon) \\ &\times K(x_2,t_m-\epsilon;x_1,t_{m-1}+\epsilon)\,K(x_1,t_{m-1}+\epsilon;x',t_{m-1}-\epsilon)\,dx_2\,dx_1, \end{aligned} \tag{36}$$

where $x''=x(t_m+\epsilon), x_2=x(t_m-\epsilon), x_1=x(t_{m-1}+\epsilon)$ and $x'=x(t_{m-1}-\epsilon)$. Then we use the symmetric action (28) to find the three propagators in the integrand and complete the integral on the right hand side of (37). Namely, we have

$$\lim_{\epsilon\to 0}K(x_2,t_m-\epsilon;x_1,t_{m-1}+\epsilon)=\lim_{\epsilon\to 0}\left[\frac{M}{2\pi i\hbar T}\right]^{1/2}\exp\left[\frac{i}{\hbar}S_0(t_m-\epsilon,t_{m-1}+\epsilon)\right], \tag{37}$$

$$\lim_{\epsilon\to 0} K(x'', t_m+\epsilon; x_2, t_m-\epsilon) \tag{38}$$
$$= \lim_{\epsilon\to 0} \left[\frac{M}{2\pi i\hbar T}\right]^{1/2} \exp\left[\frac{i}{\hbar}S_0(t_m+\epsilon, t_m-\epsilon)\right] \exp\left[-\frac{i}{4\hbar}M\omega^2 T x_m^2\right]$$
$$= \lim_{\epsilon\to 0} \delta(x_m - x_2) \exp\left[-\frac{i}{4\hbar}M\omega^2 T x_m^2\right],$$

and

$$\lim_{\epsilon\to 0} K(x_1, t_{m-1}+\epsilon; x', t_{m-1}-\epsilon) \tag{39}$$
$$= \lim_{\epsilon\to 0} \left[\frac{M}{2\pi i\hbar T}\right]^{1/2} \exp\left[\frac{i}{\hbar}S_0(t_{m-1}+\epsilon, t_{m-1}-\epsilon)\right] \exp\left[-\frac{i}{4\hbar}M\omega^2 T x_{m-1}^2\right]$$
$$= \lim_{\epsilon\to 0} \delta(x_1 - x_{m-1}) \exp\left[-\frac{i}{4\hbar}M\omega^2 T x_{m-1}^2\right].$$

In the above, we have used the relation,

$$\lim_{\epsilon\to 0}[a/\epsilon]^{1/2}\, e^{-(a/\epsilon)(x-x')^2} = \delta(x-x').$$

Substituting these results into (37), we get

$$K(x_m, x_{m-1}; T) = \left[\frac{M}{2\pi i\hbar T}\right]^{1/2} \exp\left[\frac{i}{\hbar}S_0(t_m, t_{m-1})\right] \exp\left[-\frac{1}{4\hbar}M\omega^2 T(x_m^2 + x_{m-1}^2)\right], \tag{40}$$

or

$$K(x_m, x_{m-1}; T) = \left[\frac{M}{2\pi i\hbar T}\right]^{1/2} \exp\left[\frac{i}{\hbar}\bar{S}(t_m, t_{m-1})\right], \tag{41}$$

where $S_0(t_m, t_{m-1})$ in (42) and $\bar{S}(t_m, t_{m-1})$ in (refKm2) are the free particle action (28) and the symmetric one-period action (29), respectively.

The propagator (37) evaluated from $t_{m-1}+\epsilon$ to $t_m-\epsilon$ is in fact the free particle propagator:

$$K^{free}(x_m, x_{m-1}; T) = \left[\frac{M}{2\pi i\hbar T}\right]^{1/2} \exp\left[\frac{i}{\hbar}S_0(t_m, t_{m-1})\right] = \left[\frac{M}{2\pi i\hbar T}\right]^{1/2} \exp\left[\frac{iM}{2\hbar T}(x_m - x_{m-1})^2\right]. \tag{42}$$

The state function $\psi(x_m)$ at the mth pulse is to be determined from the state $\psi(x_{m-1})$ at the $(m-1)$th pulse by

$$\psi(x_m) = \int_{-\infty}^{\infty} K(x_m, x_{m-1}; T)\, \psi(x_{m-1})\, dx_{m-1}, \tag{43}$$

and the propagator for a double period $2T$ can be found by convolution,

$$K(x_{m+1}, x_{m-1}; 2T) = \int K(x_{m+1}, x_m; T) K(x_m, x_{m-1}; T)\, dx_m. \tag{44}$$

In finding the two-period propagator via (44), we utilize the idea of harmonic analysis to expand the one-period propagator in a series of the orthogonal polynomials and to carry out the convolution with the aid of the orthogonality property of the polynomials. For this purpose, it is convenient to rewrite the symmetric one-period action (29) as

$$\bar{S}(x_m, x_{m-1}) = \frac{M}{2T}\left(1 - \frac{1}{2}\omega^2 T^2\right)\left(x_m^2 + x_{m-1}^2\right) - \frac{M}{T}(x_m x_{m-1}). \tag{45}$$

If we let

$$\cos\varphi = 1 - \omega^2 T^2/2, \quad \text{or} \quad \sin(\varphi/2) = \omega T/2 \tag{46}$$

and

$$\xi = \alpha x \quad \text{with} \quad \alpha = \sqrt{(M/\hbar T)\sin\varphi}, \tag{47}$$

the one-period action (45) may be expressed as

$$\bar{S}(x_m, x_{m-1}) = \frac{1}{2}\hbar\cot\varphi\,(\xi_m^2 + \xi_{m-1}^2) - \hbar\csc\varphi\,\xi_m\xi_{m-1}. \tag{48}$$

At this point, we relate φ to the deformation parameter q by

$$\varphi = i\ln q \tag{49}$$

as has been mentioned at the end of the previous section. Then we may express the one-period propagator for the action (48) as

$$\begin{aligned} K(x_m, x_{m-1}; T) = \; & \left[\tfrac{M}{2\pi i\hbar T}\right]^{1/2} \exp\left[-\frac{1}{2}(\xi_m^2 + \xi_{m-1}^2)\right] \\ & \times \exp\left[\frac{2\xi_m\xi_{m-1}q - (\xi_m^2 + \xi_{m-1}^2)q^2}{1 - q^2}\right]. \end{aligned} \tag{50}$$

Now we use Mehler's formula for the Hermite polynomials $H_n(x)$ (see, e.g., [9]),

$$(1 - q^2)^{-1/2}\exp\left[\frac{2xyq - (x^2 + y^2)q^2}{1 - q^2}\right] = \sum_{k=0}^{\infty}\frac{q^k H_k(x)H_k(y)}{2^k k!}, \tag{51}$$

to put the propagator (51) in the series form,

$$K(x_m, x_{m-1}; T) = \left[\frac{\alpha}{\pi}\right]^{1/2}\exp\left[-\frac{1}{2}(\xi_m^2 + \xi_{m-1}^2)\right]\sum_{k=0}^{\infty}\frac{1}{2^k k!}q^{(k+\frac{1}{2})}H_k(\xi_m)H_k(\xi_{m-1}). \tag{52}$$

Substituting this into the integrand of (44) and performing the integration with the help of the orthogonality relation for the Hermite polynomials,

$$\int_{-\infty}^{\infty} e^{-\xi^2}H_k(\xi)H_{k'}(\xi)\,d\xi = 2^k k!\sqrt{\pi}\,\delta_{k,k'}, \tag{53}$$

we arrive at the double period propagator,

$$K(x_{m+1}, x_{m-1}; 2T) = \left[\frac{\alpha}{\pi}\right]^{1/2}\exp\left[-\frac{1}{2}(\xi_{m+1}^2 + \xi_{m-1}^2)\right]\sum_{k=0}^{\infty}\frac{1}{2^k k!}q^{2(k+\frac{1}{2})}H_k(\xi_{m+1})H_k(\xi_{m-1}). \tag{54}$$

Similarly, the n-period propagator with $m=1$ is given by

$$K(x_n, x_0; nT) = \left[\frac{\alpha}{\pi}\right]^{1/2} \exp\left[-\frac{1}{2}(\xi_n^2+\xi_0^2)\right] \sum_{k=0}^{\infty} \frac{1}{2^k k!} q^{n(k+\frac{1}{2})} H_k(\xi_n) H_k(\xi_0). \tag{55}$$

This can be easily proven by induction with the help of the convolution formula (44) and the orthogonality relation (53). Now it is important to note that the n-period propagator is characterized by the nth power of the deformation parameter q.

Again, using the expansion formula (51) and noticing that $\alpha = [M\sin\varphi/\hbar T]^{1/2} = [2M(q-q^{-1})/i\hbar T]$, we put the series solution (55) back to a closed form expression:

$$\begin{aligned} K(x_n, x_0; nT) &= \left[\frac{M(q-q^{-1})}{2\pi i\hbar T(q^n-q^{-n})}\right]^{1/2} \\ &\times \exp\left[\frac{iM(q-q^{-1})}{4\hbar T(q^n-q^{-n})}\{(x_n^2+x_0^2)(q^n+q^{-n})-4x_n x_0\}\right]. \end{aligned} \tag{56}$$

If we wish to start counting pulses n times after the mth pulse, we should replace x_0 by x_m and x_n by x_{m+n} in (55). The corresponding n-period propagator is

$$\begin{aligned} K(x_{m+n}, x_m; nT) &= \left[\frac{M(q-q^{-1})}{2\pi i\hbar T(q^n-q^{-n})}\right]^{1/2} \\ &\times \exp\left[\frac{iM(q-q^{-1})}{4\hbar T(q^n-q^{-n})}\{(x_{m+n}^2+x_m^2)(q^n+q^{-n})-4x_{m+n} x_m\}\right]. \end{aligned} \tag{57}$$

The q^n dependence of the n-period propagator remains the same under the shift of the initial position.

The n-period propagator (57) or (58) is the q- representation of the propagator for the p-oscillator (or more generally for the q-object). With $q = e^{-i\varphi}$, we can express the propagator (57) in the φ-dependent form:

$$\begin{aligned} K(x_n, x_0; nT) &= \left[\frac{M}{2\pi i\hbar T U_{n-1}[\cos\varphi]}\right]^{1/2} \\ & \exp\left[\frac{iM}{2\hbar T U_{n-1}[\cos\varphi]}\{(x_n^2+x_0^2)T_n[\cos\varphi]-2x_n x_0\}\right], \end{aligned} \tag{58}$$

where $T_n[\cos\varphi]$ and $U_n[\cos\varphi]$ are the Chebyshev polynomials given in (24).

For the proper p-oscillator, φ has to be real as is in case (i). However, extending the propagator (57) or (59) by analytic continuation under the condition $q+q^{-1} \in \mathbf{R}$ to include cases (ii) and (iii), we interpret it as the propagator for the q-object.

6 Special Cases

Now we wish to analyze the propagator for the q-object for special cases.

Pulsed Harmonic Oscillator

The n-period propagator for the proper p-oscillator follows immediately from the φ-dependent propagator (59); namely,

$$\begin{aligned} K(x_n, x_0; nT) &= \left\{ \frac{M}{2\pi i\hbar T U_{n-1}[\cos\varphi(T)]} \right\}^{1/2} \\ &\times \exp\left\{ \frac{iM}{2\hbar T U_{n-1}[\cos\varphi(T)]} \{(x_n^2 + x_0^2) T_n[\cos\varphi(T)] - 2x_n x_0\} \right\}, \end{aligned} \tag{59}$$

or in terms of the sinusoidal functions,

$$\begin{aligned} K(x_n, x_0; nT) &= \left\{ \frac{M \sin[\varphi(T)]}{2\pi i\hbar T \sin[n\varphi(T)]} \right\}^{1/2} \\ &\times \exp\left\{ \frac{iM \sin[\varphi(T)]}{2\hbar T \sin[n\varphi(T)]} \{(x_n^2 + x_0^2) \cos[n\varphi(T)] - 2x_n x_0\} \right\}. \end{aligned} \tag{60}$$

In this case, the angle φ must inevitably be related to the period T by $\varphi(T) = \cos^{-1}(1 - \omega^2 T^2/2)$ under the condition $0 < \omega^2 T^2 < 4$. The corresponding deformation parameter is $q = e^{-i\varphi} = \exp[-i(1 - \omega^2 T^2/2)] \in \mathbf{R}$.

Caustics of the Pulsed Oscillator

It is evident that the zeros of $U_{n-1}(\cos\varphi)$ in the pre-factor lead the propagator (59) to divergence. The zeros occur only when $n\varphi = k\pi$ $(k = 0, \pm 1, \pm 2, ...)$; that is, only for the real p-oscillator with frequency ω meeting the restriction $0 < \omega^2 T^2 < 4$. In other words, $q^n = e^{-ik\pi}$ with $k \in \mathbf{Z}$ corresponds to the caustics of the propagator for the proper p-oscillator (60).

The Harmonic Oscillator Limit

In the limit of the zero period ($T \to 0$), the pulsed action of Hooke's force on the system becomes a continuous influence of the harmonic oscillator potential on the particle. Thus, the propagator of the standard harmonic oscillator should result from the n-period propagator (57) by the limiting process: $T \to 0$ and $n \to \infty$ such that the total time interval $nT = \tau$ remains constant. In this limit,

$$\varphi = \sin^{-1}[\omega T(1 - \omega^2 T^2/4)^{1/2}] \quad \to \quad \omega T.$$

Hence $q \to e^{-i\omega T} \to 1$ but $q^n \to e^{-i\omega\tau} \neq 1$. Consequently, the n-period propagator for the p-oscillator (61) approaches the standard result for the harmonic oscillator propagator:

$$K(x'', x'; \tau) = \sqrt{\frac{M\omega}{2\pi i\hbar \sin(\omega\tau)}} \exp\left[\frac{iM\omega}{2\hbar \sin(\omega\tau)} \{(x''^2 + x'^2)\cos(\omega\tau) - 2x'x''\} \right]. \tag{61}$$

where $x'' = x_\infty$, $x' = x_0$ and $\tau = nT$. Apparently the caustics of the p-oscillator remain to be those of the harmonic oscillator.

q-Free Particle:

To extract the n-period propagator for the T-evolution of the q-free particle from (57), we first remind ourselves that the coordinate variable y_m of the q-free particle is related to the variable x_m satisfying the Chebyshev recursion relation (20) by $x_m = q^{-m} y_m$. Thus, converting x_m into y_m, we obtain

$$\begin{aligned} K(y_{m+n}, y_m; nT) &= \left\{ \frac{M(q-q^{-1})}{2\pi i\hbar T(q^n - q^{-n})} \right\}^{1/2} \\ &\times \exp\left\{ \frac{iM(q-q^{-1})}{4\hbar T(q^n - q^{-n})} \{(q^{-2n} y_{m+n}^2 + y_m^2)(q^n + q^{-n}) - 4q^{-n} y_{m+n} y_m\} \right\} \end{aligned} \tag{62}$$

with $q \in \mathbf{R}$. Here $y_m = y(q^{2m} s_0)$ with the initial value of the time parameter $s_0 = q^{2t_0/T}$.

There are two types of q-free particles under consideration, that is, those for $q \in \mathbf{R}^+$ and for $q \in \mathbf{R}^-$. The former has real trajectories, while the latter has complex trajectories but retains real values at $t = nT$. Since we are dealing with time-evolution by $t = nT$ (which corresponds to q-progression by q^{2n}, the propagator (63) is valid for both types.

For the q-free particle of the first type ($\omega^2 T^2 < 0$), we let $q = e^{-\chi} \in \mathbf{R}^+$ with $\chi = \cosh^{-1}(1 + |\omega|^2 T^2/2) \in \mathbf{R}$, so that (63) takes the form,

$$\begin{aligned} K(y_{m+n}, y_m; nT) &= \left\{ \frac{M}{2\pi i\hbar T\, U_{n-1}[\cosh\chi]} \right\}^{1/2} \\ &\times \exp\left\{ \frac{iM}{2\hbar T\, U_{n-1}[\cosh\chi]} \{(e^{2\chi} y_{m+n}^2 + y_m^2) T_n[\cosh\chi] - 2e^{\chi} y_{m+n} y_m\} \right\}, \end{aligned} \tag{63}$$

For the q-free particle of the second type ($4 < \omega^2 T^2$), we let $q = -e^{-\chi} \in \mathbf{R}^-$ with $\chi = \cosh^{-1}(\omega^2 T^2 - 1) \in \mathbf{R}$. Then we have

$$\begin{aligned} K(y_{m+n}, y_m; nT) &= \left\{ \frac{i(-1)^n M}{2\pi\hbar T\, U_{n-1}[\cosh\chi]} \right\}^{1/2} \\ &\times \exp\left\{ -\frac{iM}{2\hbar T\, U_{n-1}[\cosh\chi]} \{(e^{2\chi} y_{m+n}^2 + y_m^2) T_n[\cosh\chi] - 2e^{\chi} y_{m+n} y_m\} \right\}, \end{aligned} \tag{64}$$

Newtonian Free Particle:

As the deformation parameter q tends to unity (or $\omega \to 0$), the deformed number $[n] = (q^n - q^{-n})/(q - q^{-1})$ approaches n. Hence the propagators (57) and (63) both reduce to the standard free propagator:

$$K(x_n, x_0; nT) = \left[\frac{M}{2\pi i\hbar nT} \right]^{1/2} \exp\left\{ \frac{iM}{2\hbar nT} (x_n - x_0)^2 \right\}. \tag{65}$$

This is an expected result.

7 Conclusion

We have observed the similarity between the deformed free particle and the pulsed harmonic oscillator via their recursion relation. By using this interesting nature, we have treated the two systems as special cases of a single q-deformed system (q-object) and evaluated the propagator for the Chebyshev process. From this unified treatment, we have been able to derive the n-period propagator for the T-evolution of the q-free particle as well as that of the p-oscillator. The boson limit $q = 1$ gives the ordinary free particle propagator. While $q \in \mathbf{R}$ for the q-free particle, $q \in S^1$ for the p-oscillator.

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